Math 6261 2023-04-14

§5. Martingales

§5.1 Conditional expectation

- Let X be a r.v. on a prob. space $(\Omega, \mathcal{F}, P)$ such that

$$E|X| < \infty.$$

Let $\mathcal{B} \subset \mathcal{F}$ be a sub-σ-algebra. Then

$$\mu(A) = \int_A X \, dP, \quad A \in \mathcal{B}$$

defines a signed measure on $(\Omega, \mathcal{B})$ and μ is absolutely continuous with respect to P, i.e. $P(A)=0 \Rightarrow \mu(A)=0$.

By the Radon-Nikodym Theorem, there is a function

$$g \in L^1(\Omega, \mathcal{B}, P)$$

such that

$$\mu(A) = \int_A g \, dP, \quad A \in \mathcal{B}.$$

Moreover, any other such function g' must coincide with g almost surely wrt μ.

Def. We use the notation $E(X|\mathcal{B})$ for g, and call $E(X|\mathcal{B})$ the conditional expectation of X with respect to $\mathcal{B}$.

- As an element of $L^1(\Omega, \mathcal{F}, P)$, $E(X|\beta)$ is completely characterized by the following two properties:

(1) $E(X|\beta)$ is β-measurable;

(2) $\int_A E(X|\beta)\,dP = \int_A X\,dP$ for all $A \in \beta$.

Example 5.1. Let $\beta = \{\phi, E, E^c, \Omega\}$ with $E \in \mathcal{F}$.

Then β is a finite sub-sigma algebra of $\mathcal{F}$.

Let $X = \mathbb{1}_A$ with $A \in \mathcal{F}$. Let $g = E(\mathbb{1}_A | \beta)$.

Since g is β-measurable, it must be constant on E and E^c.

By property (2),

$$g(x) = \begin{cases} \int_E X\,dP / P(E) & \text{if } x \in E; \\ \int_{E^c} X\,dP / P(E^c) & \text{if } x \in E^c. \end{cases}$$

Since $X = \mathbb{1}_A$,

$$g(x) = \begin{cases} P(A \cap E)/P(E) & \text{if } x \in E. \\ P(A \cap E^c)/P(E^c) & \text{if } x \in E^c. \end{cases}$$

Thus, $g(x)$ gives the prob. of the occurrence of A once we know to which element of $\mathcal{F}$ the point x belongs to.

Example 5.2 Similarly let $\mathcal{B} = \sigma(\Omega_1, \cdots, \Omega_k)$ be the σ-algebra generated by a finite partition $\Omega_1, \cdots, \Omega_k$ of Ω.

Then

$$E(X|\mathcal{B})(x) = \int_{\Omega_i} X \, dP \Big/ P(\Omega_i)$$

if $x \in \Omega_i$.

The following properties of conditional expectations can be proved easily by using the characterization by (i) and (ii):

Prop. 5.3. (1) If $X \geq 0$ a.s., then $E(X|\mathcal{B}) \geq 0$ a.s.

(2) $E(aX + bY|\mathcal{B}) = aE(X|\mathcal{B}) + bE(Y|\mathcal{B})$ a.s. for $X, Y \in L^1$, $a, b \in \mathbb{R}$.

(2) If X is $\mathcal{B}$-measurable, then $E(X|\mathcal{B}) = X$ a.s.

(3) If X is $\mathcal{B}$-measurable and $X \in L^\infty(\Omega, \mathcal{F}, P)$, then for each $Y \in L^1(\Omega, \mathcal{F}, P)$

$$E(XY|\mathcal{B}) = X \, E(Y|\mathcal{B}) \quad \text{a.s.}$$

(4) $E\big(E(X|\mathcal{B})\big) = EX$.

(5) If $\mathcal{B}_2 \subset \mathcal{B}_1$, then

$$E\big(E(X|\mathcal{B}_1)\,\big|\,\mathcal{B}_2\big) = E(X|\mathcal{B}_2) \quad \text{a.s.}$$

(6) If X is independent of $\mathcal{B}$, then

$$E(X|\mathcal{B}) = E(X) \quad \text{a.s.}$$

Pf. We prove (5), (6) only.

Let $\mathcal{B}_2 \subset \mathcal{B}_1$. Let $g = E(E(X|\mathcal{B}_1)|\mathcal{B}_2)$.

Clearly g is $\mathcal{B}_2$-measurable. Moreover, for any $A \in \mathcal{B}_2$,

$$\int_A g \, dP = \int_A E(X|\mathcal{B}_1) \, dP = \int_A X \, dP,$$

where the second equality follows from the fact that $A \in \mathcal{B}_1$.

Hence by the characterization of the conditional expectation,

$$g = E(X|\mathcal{B}_1).$$

This proves (5).

To see (6), recall that X is independent of $\mathcal{B}$, so

X and $\mathbb{1}_B$ are independent for each $B \in \mathcal{B}$

. .

Hence $$\int_B X \, dP = \int \mathbb{1}_B \cdot X \, dP = \int \mathbb{1}_B \, dP \int X \, dP$$

$$= P(B) \, E(X).$$

$$= \int_B E(X) \, dP.$$

It follows that $E(X|\mathcal{B}) = E(X)$ a.s.

Prop. 5.4. Suppose $X_n \to X$ a.s. and $|X_n| \leq Y$ and $EY < \infty$.

Then $E(X_n | \mathcal{B}) \to E(X|\mathcal{B})$ a.s.

Pf. Let $Z_n = \sup_{k \geq n} |X_k - X|$. Then $Z_n \searrow 0$ a.s. and $0 \leq Z_n \leq 2Y$

Now

$$|E(X_n|\mathcal{B}) - E(X|\mathcal{B})| = |E(X_n - X|\mathcal{B})|$$

$$\leq E(Z_n|\mathcal{B}).$$

Since Z_n is non-increasing, so is $E(Z_n|\mathcal{B})$.

Hence $E(Z_n|\mathcal{B})$ has a limit Z with $Z \geq 0$.

Notice that

$$E(Z) = \int Z \, dP \leq \int E(Z_n|\mathcal{B}) \, dP$$

$$= E(Z_n) \to 0.$$

(by the DCT)

Hence $Z \equiv 0$ a.s.

It follows that $E(Z_n|\mathcal{B}) \to 0$ a.s, and thus

$$E(X_n|\mathcal{B}) \to E(X|\mathcal{B}). \quad \square$$

Prop 5.5 (Jensen inequality)

Let $\varphi : \mathbb{R} \to \mathbb{R}$ be convex. Suppose that X and $\varphi(X)$ are integrable.

Then $\phi(E(X|\mathcal{B})) \leq E(\varphi(X)|\mathcal{B})$ a.s.

Pf. Since φ is convex, it is known that

$$\varphi'(q+) \text{ exists at each } q \in \mathbb{R}$$

and

$$\varphi(x) \geq \varphi'(q+)(x-q) + \varphi(q) \quad \text{for all } x \in \mathbb{R}.$$

Hence, we have

$$\varphi(x) = \sup_{q \in \mathbb{R}} \varphi'(q+)(x-q) + \varphi(q) \qquad \text{for all } x \in \mathbb{R}$$

Notice that $\varphi'(q)$ exists and is continuous, except for at most countable many points. It follows that $\exists$ a sequence (a_n), (b_n) in $\mathbb{R}$ such that

(1) $$\varphi(x) = \sup_n a_n x + b_n \qquad \text{for all } x \in \mathbb{R}.$$

Now for fixed n,

$$\varphi(X) \geq a_n X + b_n$$

So

$$E(\varphi(X)|\mathcal{B}) \geq a_n E(X|\mathcal{B}) + b_n \quad \text{a.s}$$

Hence $$E(\varphi(X)|\mathcal{B}) \geq \sup_n a_n E(X|\mathcal{B}) + b_n \quad \text{a.s}$$
$$= \varphi(E(X|\mathcal{B})) \quad \text{a.s.} \quad \text{(by (1))}.$$ ▨

§ 5.2 Martingales and examples.

Def. Let $(\Omega, \mathcal{F}, P)$ be a prob. space, and $\mathcal{F}_1 \subset \mathcal{F}_2 \subset \cdots$ an increasing sequence of sub-σ-algebras of $\mathcal{F}$.

A sequence $X_1, X_2, \cdots,$ of r.v.'s such that $E|X_n| < \infty$ is said to be

a submartingale if $\quad E(X_{n+1} | \mathcal{F}_n) \geq X_n$ a.s

a martingale if $\quad E(X_{n+1} | \mathcal{F}_n) = X_n$ a.s

a supermartingale if $\quad E(X_{n+1} | \mathcal{F}_n) \leq X_n$ a.s.

Remark: • the word 'martingale' was formerly used for a certain betting system.

• If X_n is the fortune of a gambler after n plays, in case of a submartingale the game is favorable to the player; in case of a supermartingale, the game is unfavorable to the player.

Examples:

(a) Sums of independent r.v.'s of zero mean.

Let $S_n = X_1 + \cdots + X_n$, where $X_1, \cdots, X_n, \cdots$ are independent r.v.'s with $E|X_n| < \infty$ and $EX_n = 0$.

Let $\mathcal{F}_n = \sigma(X_1, \cdots, X_n)$.

Then
$$\begin{aligned} E(S_{n+1} \mid \mathcal{F}_n) &= E(S_n \mid \mathcal{F}_n) + E(X_{n+1} \mid \mathcal{F}_n) \\ &= S_n + E(X_{n+1} \mid \mathcal{F}_n) \\ &\quad (\text{since } S_n \text{ is } \mathcal{F}_n\text{-measurable}) \\ &= S_n + E(X_{n+1}) \quad (\text{since } X_{n+1} \text{ is independent of } \mathcal{F}_n) \\ &= S_n. \end{aligned}$$

(b) Products of non-negative independent r.v.'s of mean 1.

Let $X_1, X_2, \cdots$ be non-negative independent r.v.'s with $E(X_n) = 1$.

Let $M_n = X_1 \cdots X_n$.

Then for $n \geq 1$,

$$\begin{aligned} E(M_{n+1} \mid \mathcal{F}_n) &= E(X_{n+1} M_n \mid \mathcal{F}_n) \\ &= M_n E(X_{n+1} \mid \mathcal{F}_n) \quad (\text{since } M_n \text{ is } \mathcal{F}_n\text{-measurable}) \\ &= M_n E(X_{n+1}) = M_n. \end{aligned}$$

(since X_{n+1} and $\mathcal{F}_n$ are independent)

So (M_n) is a martingale.